Original Article

N=4 Mechanics, WDVV Equations and Polytopes[a]

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Abstract

N=4 superconformal n-particle quantum mechanics on the real line is governed by two prepotentials, *U* and *F*, which obey a system of partial nonlinear differential equations generalizing the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation for *F*. The solutions are encoded by the finite Coxeter systems and certain deformations thereof, which can be encoded by particular polytopes. We provide A_n and B_3 examples in some detail. Turning on the prepotential *U* in a given *F* background is very constrained for more than three particles and nonzero central charge. The standard ansatz for *U* is shown to fail for all finite Coxeter systems. Three-particle models are more flexible and based on the dihedral root systems.

Conformal Quantum Mechanics: Calogero System

We are investigating systems of n+1 identical point particles with unit mass whose motion on the real line is governed by the Hamiltonian

$$H = \frac{1}{2} p_i p_i + V_B(x^1,...,x^{n+1})$$ (1.1)

and subject to the canonical quantization relations

$$[x^i, p_j] = i\delta^i_j.$$ (1.2)

together with

$$D = -\frac{1}{4}(x^i p_i + p_i x^i) \text{ and } K = \frac{1}{2} x^i x^i,$$ (1.3)

this Hamiltonian realizes an so(1, 2) conformal algebra

$$[D,H] = -iH, \ [H,K] = 2iD, \ [D,K] = iK$$ (1.4)

if the potential is homogeneous of degree −2,.

$$(X^i \partial_i + 2) \ V_B = 0.$$ (1.5)

When demanding also permutation and translation invariance as well as admitting only two-body forces, the solution is uniquely given by the Calogero potential,

$$V_B = \sum_{i<j} \frac{g^2}{\left(x^i - x^j\right)^2}$$ (1.6)

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N = 4 Super Conformal Extension: Su (1, 1|2) Algebra

Let us extend the algebra from $so(1,2) \oplus su(1,1)$ to the super algebra $su(1,1|2)$ with central charge C by enlarging the set of generators

$$(H,D,K) \to (H,D,K,Q_\alpha,S_\alpha,J_a,C) \text{ With } \alpha = 1,2, \ a = 1,2,3, \ (Q_\alpha)^\dagger = \overline{Q}^\alpha, \ (S_\alpha)^\dagger = \overline{S}^\alpha$$

And imposing the nonvanishing (anti)commutators:

$$[D,H] = -iH \qquad\qquad [D,K] = +iK$$

$$[D,K] = +iK \qquad\qquad [J_a,J_b] = i \in_{abc} J_c$$

$$\{Q_\alpha,\overline{Q}^\beta\} = 2H\delta_\alpha^{\ \beta} \qquad\qquad \{Q_\alpha \overline{S}^\beta\} = +2i(\sigma_a)_\alpha^{\ \beta} J_a - 2D\delta_\alpha^{\ \beta} - iC\delta_\alpha^{\ \beta}$$

$$\{S_\alpha,\overline{S}^\beta\} = 2K\delta_\alpha^{\ \beta} \qquad\qquad \{\overline{Q}^\alpha,S_\beta\} = -2i(\alpha_a)_\beta^{\ \alpha} J_a - 2D\delta_\beta^{\ \alpha} + iC\delta_\beta^{\ \alpha}$$

$$[D,Q_\alpha] = -\frac{i}{2}Q_\alpha \qquad\qquad [D,S_\alpha] = +\frac{i}{2}S_\alpha$$

$$[K,Q_\alpha] = +iS_\alpha \qquad\qquad [H,S_\alpha] = -iQ_\alpha$$

$$[J_a,Q_\alpha] = -\frac{1}{2}(\sigma_a)_\alpha^{\ \beta}Q_\beta \qquad\qquad [J_a,S_\alpha] = -\frac{1}{2}(\sigma_a)_\alpha^{\ \beta}S_\beta$$

$$[D,\overline{Q}^\alpha] = -\frac{i}{2}\overline{Q}^\alpha \qquad\qquad [D,\overline{S}^\alpha] = +\frac{i}{2}\overline{S}^\alpha$$

$$[K,\overline{Q}^\alpha] = +i\overline{S}^\alpha \qquad\qquad [H,\overline{S}^\alpha] = -i\overline{Q}^\alpha$$

$$[J_a,\overline{Q}^\alpha] = \frac{1}{2}\overline{Q}^\beta(\sigma_a)_\beta^{\ \alpha} \qquad\qquad [J_a,\overline{S}^\alpha] = \frac{1}{2}\overline{S}^\beta(\sigma_a)_\beta^{\ \alpha}$$

To realize this algebra one must pair the bosonic coordinates x^i with fermionic partners ψ^i_α and $\overline{\psi}^{i\alpha} = \psi^{i\ \dagger}_\alpha$ with $i = 1, \ldots, n+1$ and $\alpha = 1, 2$ subject to

$$\{\psi^i_{\ \alpha},\psi^j_{\ \beta}\} = 0, \qquad \{\overline{\psi}^{i\alpha},\overline{\psi}^{i\beta}\} = 0, \qquad \{\psi^i_{\ \alpha},\overline{\psi}^{j\beta}\} = \delta_\alpha^{\ \beta}\delta^{ij}. \tag{2.1}$$

Surprisingly, the non-interacting generator candidates

$$Q_{0\alpha} = pi\psi^i_{\ \alpha}, \ \overline{Q}_0^\alpha = pi\overline{\psi}^{i\alpha} \text{ and } S_{0\alpha} = x^i\psi^i_{\ \alpha}, \ \overline{S}_0^\alpha = x^i\overline{\psi}^{i\alpha}, \tag{2.2}$$

$$H_0 = \frac{1}{2}p_ip_i, \ D_0 = -\frac{1}{4}(x^ip_i + p_ix^i), \ K_0 = \frac{1}{2}x^ix^i, \ J_{0a} = \frac{1}{2}\overline{\psi}^{i\alpha}(\sigma_a)_\alpha^{\ \beta}\psi^i_\beta \tag{2.3}$$

fail to obey the *su*(1, 1|2) algebra, and hence interactions are needed! Their simplest implementation changes only

$$Q_\alpha = Q_{0\alpha} - i\,[S_{0\alpha},V] \text{ and } H = H_0 + V, \tag{2.4}$$

Just requiring the invention of a potential $V(x,\psi,\overline{\psi})$.

A minimal ansatz to close the *su*(1, 1|2) algebra reads [1,2]

$$V = V_B(x) - U_{ij}(x)\left\langle \psi^i_\alpha \overline{\psi}^{j\alpha} \right\rangle + \frac{1}{4}F_{ijkl}(x)\left\langle \psi^i_\alpha \psi^{j\alpha} \overline{\psi}^{k\beta}\overline{\psi}^l_{\ \beta} \right\rangle \tag{2.5}$$

Where $\left\langle \ldots \right\rangle$ denotes symmetric (Weyl) ordering. The coefficient functions U_{ij} and F_{ijkl} are totally symmetric and homogeneous of degree -2. With this, the super symmetry generators in (2.4) become

$$Q_\alpha = (pj - ix^iU_{ij}(x)) \ \psi^j_\alpha - \frac{i}{2}x^iF_{ijkl}(x)\left\langle \psi^j_\beta\psi^{k\beta}\overline{\psi}^l_\alpha \right\rangle. \tag{2.6}$$

The Structure Equations: WDVV, Flatness, Homogeneity

Inserting the minimal V ansatz (2.5) into the *su*(1, 1|2) algebra and demanding its closure produces conditions on U_{ij} and F_{ijkl} First, one learns that

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$$U_{ij} = \partial_i \partial_j U \text{ and } F_{ijkl} = \partial_i \partial_j \partial_k \partial_l F, \tag{3.1}$$

introducing two scalar prepotentials U and F. Second, these prepotentials are subject to the "structure equations" [1,2]

$$(\partial_i \partial_k \partial_p F)(\partial_p \partial_l \partial_j F) = (\partial_i \partial_l \partial_p F), \qquad x^i \partial_i \partial_j \partial_k F = -\delta_{jk} \tag{3.2}$$

$$\partial_i \partial_j U - (\partial_i \partial_j \partial_k F) \partial_k U = 0, \qquad x^i \partial_i U = -C. \tag{3.3}$$

The quadratic equation for F is the famous WDVV equation [3,4]. The relation below it (linear in U) resembles a covariant constancy equation, and we label it as the "flatness condition". Its consistency implies the WDVV equation contracted with $\partial_j U$. Both the WDVV equation and the flatness condition trivialize when contracted with x^i. Finally, the two right equations are homogeneity properties for F and U. One of their consequences is

$$x^i F_{ijkl} = -\partial_j \partial_k \partial_l F \text{ and } x^i U_{ij} = -\partial_j U. \tag{3.4}$$

The one for F may be integrated twice to

$$(x^i \partial_i - 2)F = -\frac{1}{2} x^i x^i. \tag{3.5}$$

Clearly, there is the redundancy of adding a quadratic polynomial to F and a constant to U. The third outcome of the $su(1, 1|2)$ algebra is

$$V_B = \frac{1}{2}(\partial_i U)(\partial_i U) + \frac{\hbar^2}{8}(\partial_i \partial_j \partial_k F)(\partial_i \partial_j \partial_k F) \tag{3.6}$$

Where we have reinstalled ~ to exhibit the quantum part in V_B.

In case of vanishing central charge, $C = 0$, a partial solution consists in putting $U \equiv 0$. Since U does not enter in (3.2), the natural strategy is to firstly solve the WDVV equation and secondly turn on a flat U in this F background.

Prepotential Ansatz: Covectors and Couplings

The homogeneity conditions $(x^i \partial_i - 2)F = -\frac{1}{2} x^i x^i$ und $x^i \partial_i U = -C$ are solved by [1]

$$F = -\frac{1}{2} \sum_\alpha f_\alpha \alpha(x)^2 \ln |\alpha(x)| + F_{\text{hom}} \text{ and } U = -\sum_\alpha g_\alpha \ln |\alpha(x)| + U_{\text{hom}}, \tag{4.1}$$

Where F_{hom} and U_{hom} are arbitrary homogeneous functions of degree −2 and 0, respectively. The sums run over a set of real covectors α (not indexed!) with values $\alpha(x) = \alpha_i x^i$, which are subject to the constraints

$$\sum_\alpha f_\alpha \alpha(x^2) = x^i x^i =: R^2 \text{ and } \sum_\alpha g_\alpha = C. \tag{4.2}$$

The coefficients f_α are essentially fixed by (4.2) and (if positive) may be absorbed into a rescaling of α, while the g_α will emerge as coupling constants which, however, may be frozen to zero. One may rewrite the expressions (4.1) as

$$F = -\frac{1}{2} R^2 \ln R + F_{\text{hom}} \qquad \text{and} \qquad U = -C \ln R + U'_{\text{hom}} \tag{4.3}$$

or linearly combine (4.1) and (4.3) with coefficents adding to one.

Due to the generality of F_{hom}, we are currently unable to solve the WDVV equation (3.2) with (4.1) or (4.3), except for $F_{\text{hom}}^{(')} = 0$. Even then, the nonlinearity of (3.2) restricts the linear combinations to

$$F = -\frac{1}{2} \sum_\alpha f \alpha(x)^2 \ln |\alpha(x)| \qquad \text{or} \qquad F = +\frac{1}{2} \sum_\alpha f_\alpha \alpha(x)^2 \ln |\alpha(x)| - R^2 \ln R, \tag{4.4}$$

and imposes [5,6]

$$\sum_{\alpha, \beta} f_\alpha f_\beta \frac{\alpha \cdot \beta}{\alpha(x)\beta(x)} (\alpha \wedge \beta)^{\otimes 2} = 0 \quad \text{with} \quad (\alpha \wedge \beta)_{ijkl}^{\otimes 2} = (\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_k \beta_l - \alpha_l \beta_k). \tag{4.5}$$

Thus, let us limit ourselves to the ansatz (4.4) and try to turn on U. Even this is too difficult in general, so let us drop the homogeneous pieces in (4.1) and (4.3) and just combine the inhomogeneous parts. Then, the flatness condition (3.3) rules out all 'R' terms in F or U and demands

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$$\sum_{\beta} \left(g_{\beta} \frac{1}{\beta(x)} - f_{\beta} \sum_{\alpha} g_{\alpha} \frac{\alpha \cdot \beta}{\alpha(x)} \right) \frac{1}{\beta(x)} \beta \otimes \beta = 0 \text{ with } (\beta \otimes \beta)_{ij} = \beta_i \beta_j,$$

(4.6)

While the bosonic potential reads

$$V_B = \frac{1}{2} \sum_{\alpha, \beta} \frac{\alpha \cdot \beta}{\alpha(x)\beta(x)} (g_{\alpha} g_{\beta} + \frac{\hbar^2}{4} f_{\alpha} f_{\beta} (\alpha \cdot \beta)^2).$$

(4.7)

Because the equations decouple for mutually orthogonal sets of covectors, it suffices to take $\{\alpha\}$ as being indecomposable. In particular, it is convenient in translation-invariant models to decouple the center of mass $\alpha_{com}(x) = \sum_i x^i$, reducing the bosonic configuration space from $\square^{n+1}$ to $\square^n$. Note that this alters

$$R^2 = x^i x^i \quad \text{to} \quad R^2 = \left(x^i - \frac{1}{n+1} \alpha_{com}(x) \right)^2 = \frac{1}{n+1} [n \sum_i x^i x^i - 2 \sum_{j<l} x^k x^l].$$

(4.8)

Partial results are known for n ≤ 3 [1,2,7-10], but the case n = 2 is special since the WDVV equation is empty then, which admits many extra solutions.

U = 0 Solutions: Root Systems

It was shown by Martini and Gragert [5] and extended by Veselov [6] that the set Φ+ of positive roots of any simple Lie algebra solves the left equation in (4.5). Let us normalize the long and short roots as

$$\alpha \cdot \alpha = 2 \text{ for } \alpha \in \phi_L^+ \text{ and } \beta \cdot \beta = \frac{2}{r} \text{ for } \beta \in \phi_S^+ \text{ with } r = 1, 2, 3.$$

(5.1)

Recalling that

$$\sum_{\alpha \in \phi^+} \alpha \otimes \alpha = h^{\vee} 1 \text{ and } \sum_{\alpha \in \phi^+} \frac{2}{\alpha \cdot \alpha} \alpha \otimes \alpha = h 1$$

(5.2)

are determined by the Coxeter number h and the dual Coxeter number $h^{\vee}$, the left condition in (4.2) becomes

$$1 = \sum_{\alpha} f_{\alpha} \alpha \otimes \alpha = f_L \sum_{\alpha \in \phi_L^+} \alpha \otimes \alpha + fs \sum_{\alpha \in \phi^+} \alpha \otimes \alpha = f_L \frac{rh^{\vee} - h}{r-1} 1 + f_s \frac{h - h^{\vee}}{r-1} 1,$$

(5.3)

which is solved by the one-parameter family

$$f_L = \frac{1}{h^{\vee}} + \left(\frac{h}{h^{\vee}} - 1 \right) t \text{ and } f_s = \frac{1}{h^{\vee}} - (r - \frac{h}{h^{\vee}}) t \text{ for } t \in \square.$$

(5.4)

It is not hard to see that the subset of roots belonging to any plane spanned by a short root β and its string $(\alpha, \alpha + \beta, \alpha + 2\beta, ..., \alpha + r\beta)$ through a long root α makes the double sum in the left equation of (4.5) already vanish. Since the full double sum decomposes into contributions of such planes, we get a prepotential solution (Figure 1)

$$F(t) = \frac{1}{2} (f_L \sum_{\alpha \in \phi_L^+} + f_S \sum_{\alpha \in \phi_S^+}) \alpha(x)^2 \ln |\alpha(x)|$$

(5.5)

$$= -\frac{1}{2h^{\vee}} \sum_{\alpha \in \phi^+} \alpha(x)^2 \ln |\alpha(x)| - \frac{t}{2} \left[\left(\frac{h}{h^{\vee}} - 1 \right) \sum_{\alpha \in \phi_L^+} - (r - \frac{h}{h^{\vee}}) \sum_{\alpha \in \phi_S^+} \right] \alpha(x)^2 \ln |\alpha(x)|$$
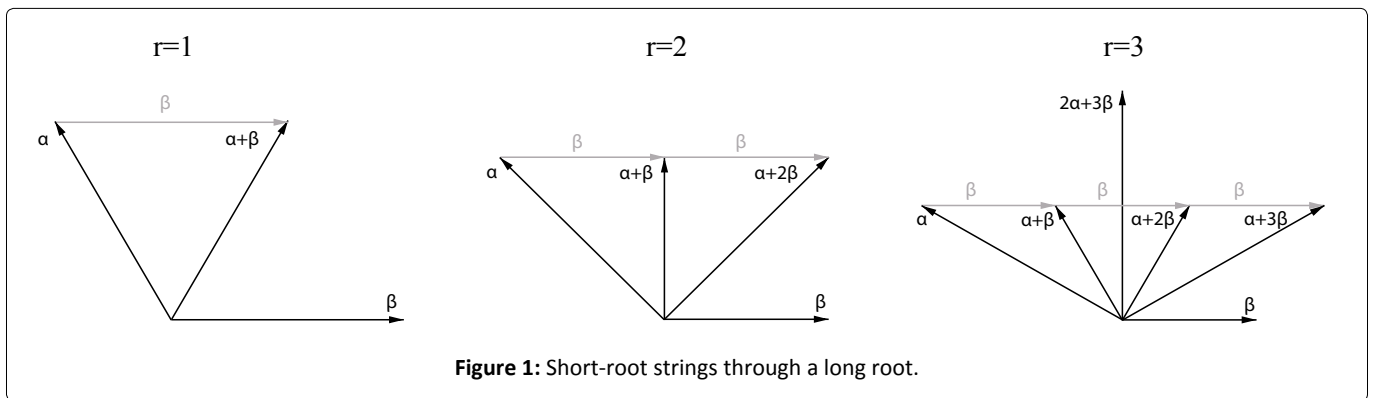

Figure 1: Short-root strings through a long root.

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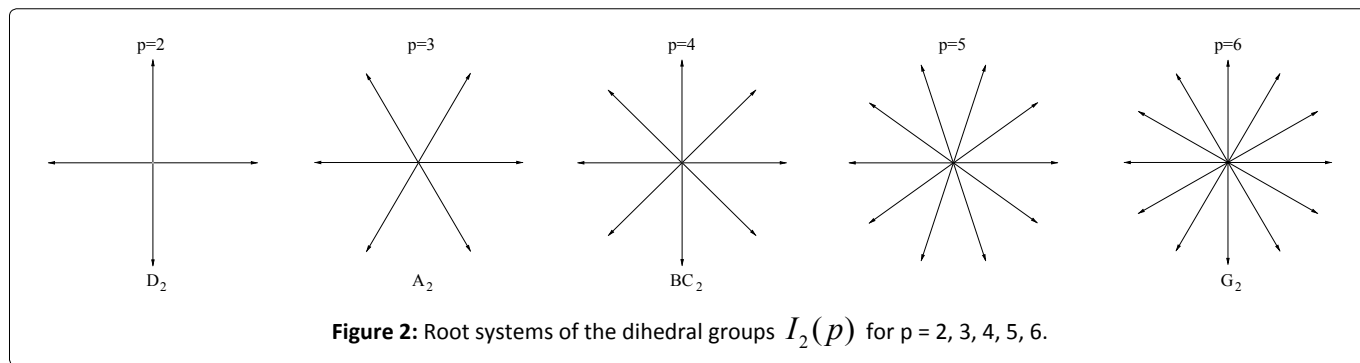
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Figure 2: Root systems of the dihedral groups $I_2(p)$ for p = 2, 3, 4, 5, 6.

which is unique only for simply-laced Lie algebras. Note that $|f_L|$ and $|f_S|$ may be absorbed into a rescaling of α but their signs cannot, and so the non-simply-laced solution generalizes the $t = 0$ one found before [5,6] by adding to it a concrete F_{hom}.

Let us give two examples, with n + 1 and 3 particles, respectively:

$$A_n \oplus A_1 : \{\alpha(x)\} = \{x^i - x^j, \ \textstyle\sum_i x^i \mid 1 \le i < j \le n+1\} \text{ and } f_\alpha = \frac{1}{n+1}, \tag{5.6}$$

$$G_2 \oplus A_1 : \{\alpha(x)\} = \left\{ \frac{1}{\sqrt{3}}(x^i + x^j - 2x^k), \ \frac{1}{\sqrt{3}}(x^i - x^j), x^1 + x^2 + x^3 \mid (i, j, k) \text{ cyclic} \right\} \tag{5.7}$$

and $f_L = \frac{1}{4} + \frac{1}{2}t, f_s = \frac{1}{4} - \frac{3}{2}t, f_{com} = \frac{1}{3}.$

The Weyl groups of the simple Lie algebras can be extended by the non-crystallographic Coxeter groups H_4 (60 positive roots), H_3 (15 positive roots) and $I_2(p)$ (p positive roots), which also clear the WDVV equation [6]. The dihedral groups $I_2(p)$ with $f_\alpha = \frac{1}{p}$ cover all rank-two root systems, including $A_1 \otimes A_1, A_2, BC_2$ and G_2 for p = 2, 3, 4 and 6, respectively, upon rescaling of α (Figure 2).

U = 0 Solutions: Deformed Root Systems

The Lie-algebra root systems are only the tip of an iceberg of WDVV solutions. It has been shown [7,9] that certain deformations of them retain the WDVV property. Let us rephrase some examples in our terminology.

The three positive roots of A_2 may be rearranged as the edges of an equilateral triangle. Consider now a deformation of this triangle, keeping the incidence relation $\alpha + \beta - \lambda = 0$. The homogeneity condition (4.2) (and therefore also the WDVV equation) is easily solved by $f_\alpha = \left| \frac{\beta \cdot \gamma}{4A^2} \right|$ and cyclic permutations, where A denotes the area of the triangle.

If we try the same idea on the A_3 system, we obtain the six edges of a regular tetrahedron and deform to encounter the five-dimensional moduli space of tetrahedral shapes (modulo scale). Again, the homogeneity condition (4.2) has a unique solution f_α, but now the WDVV equation enforces the three conditions (Figure 3)

$$\alpha \cdot \alpha' = 0, \ \beta \cdot \beta' = 0, \ \gamma \cdot \gamma' = 0 \tag{6.1}$$

on the skew edge pairs. These relations restrict the above moduli space to the three-dimensional subspace of orthocentric tetrahedra (modulo scale), with

$$f_\alpha = \left| \frac{\beta \cdot \gamma \ \beta' \cdot \gamma'}{36V^2} \right| \text{ and } \qquad , \ \left| \frac{\beta \cdot \gamma \ \beta' \cdot \gamma}{36} \right| \text{ and } \text{ cyclic} \tag{6.2}$$

where V is the volume. Alternatively, we may implement the conditions (6.1) by picking three non-coplanar covectors, say α', β' and $\gamma',$ scaling them such that

$$\alpha' \cdot \beta' = \beta' \cdot \gamma' = \gamma' \cdot \alpha' = 1 \tag{6.3}$$

And employing the three-dimensional vector product in fixing the remaining three covectors via

$$\alpha = \alpha' \times (\beta' \times \gamma') = \beta' - \gamma', \ \beta = \beta' \times (\gamma' \times \alpha') = \gamma' - \alpha', \ \gamma = \gamma' \times (\alpha' \times \beta') = \alpha' - \beta'. \tag{6.4}$$
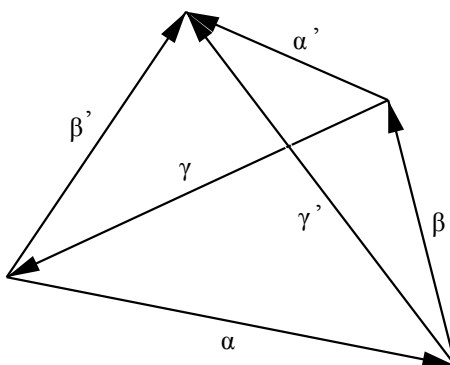
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Figure 3: Tetrahedral configuration of covectors.

Figure 4: Truncated cube.

With these data one gets $6V = \alpha' \cdot (\beta' \times \gamma')$ as well as

$$f_\alpha = \frac{\alpha' \cdot \alpha' - 1}{36V^2} \text{ and } f_{\alpha'} = \frac{(\beta' \cdot \beta' - 1)(\gamma' \cdot \gamma' - 1)}{36 \, V^2} \text{ and cyclic.} \tag{6.5}$$

In fact, this strategy generalizes to orthocentric n-simplices as n-parametric deformations of the regular n-simplex generated by the $\frac{1}{2}n(n+1)$ positive roots of A_n, with

$$f_\alpha = \frac{\left| \beta \cdot \gamma \; \beta' \cdot \gamma' \beta'' \cdot \gamma'' \dots \beta^{(n-2)} \cdot \gamma^{(n-2)} \right|}{(n! \; V)^2} \text{ etc} \tag{6.6}$$

The orthocentricity derives from the WDVV equation by a the following dimensional reduction argument. Take $\hat{n}_i x^i \to \infty$ for some fixed convector $\hat{n}$. Then, any factor $\frac{1}{\alpha(x)}$ in the WDVV equation (4.5) vanishes unless $\alpha \cdot \hat{n} = 0$, which amounts to a reduction of the covector set $\{\alpha\}$ to its intersection with the hyper plane orthogonal to $\hat{n}$. This process may be iterated until only covectors laying in a plane $\alpha \wedge \beta$ spanned by two covectors α and β survive. This situation admits two possibilities: Either the α and β are concurrent, in which case another covector α + β or α − β completes a triangle satisfying the WDVV equation, or else α and β are skew, in which case there is no further covector in their plane and WDVV demands orthogonality (Figure 4).

The B_3 root system provides another example. Four copies of the 3 short and 6 long positive roots can be assembled into the edge set of a truncated cube. We deform this polyhedron to

$$\{\alpha(x)\} = \{d_1 x^1, \, d_2 x^2, \, d_3 x^3; \; c_3(c_2 x^1 \pm c_1 x^2), \; c_1(c_3 x^2 \pm c_2 x^3), \; c_2(c_1 x^3 \pm c_3 x^1)\} \tag{6.7}$$

with $c_i, d_i \in \Box$, , retaining the 'incidence relations' of a truncated cuboid. For $c^2 := c_0^2 + c_1^2 + c_2^2 + c_3^2$ and

$$\{f_\alpha\} = \left\{ \frac{c_0^2 + c_1^2 - c_2^2 - c_3^2}{c^2 d_1^2}, \; \frac{c_0^2 - c_1^2 + c_2^2 - c_3^2}{c^2 d_1^2}, \; \frac{c_0^2 - c_1^2 - c_2^2 + c_3^2}{c^2 d_1^2}; \; \frac{1}{c^2 c_3^2}, \; \frac{1}{c^2 c_1^2}, \; \frac{1}{c^2 c_2^2} \right\} \tag{6.8}$$

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we satisfy the homogeneity condition (4.2), i.e. $\sum_\alpha f_\alpha \alpha \otimes \alpha = 1$. The relevant combinations $\sqrt{f_\alpha}\,\alpha$ depend only on

the three ratios c_i / c_0. The rigid B_3 root system with (5.4) occurs for $c_1 = c_2 = c_3 = 1$ and $c_0^2 = \dfrac{2-3t}{1+t}$. The case of C_3 is very similar.

Finally, let us present an example based on weights rather than roots, namely a deformation of the B_3 representation $\underline{7} \otimes \underline{8}$ i.e. the vector plus spin or weights. For the 3 positive 'vector' and 4 positive 'spin or' covectors we take

$$\alpha(x) = d_1 x^1, \ \ \beta(x) = d_2 x^2, \ \ \gamma(x) = d_3 x^3; \ \ \frac{\alpha+\beta+\gamma}{2}, \ \ \frac{\alpha-\beta-\gamma}{2}, \ \ \frac{-\alpha+\beta-\gamma}{2}, \ \ \frac{-\alpha-\beta+\gamma}{2} \tag{6.9}$$

with $d_i \in \square$, keeping the relations between vector and spin or weights. For $d^2 := d_1^2 + d_2^2 + d_3^2$ and

$$f_\alpha = \frac{-d_1^2 + d_2^2 + d_3^2}{d^2 d_1^2}, \ f_\beta = \frac{d_1^2 - d_2^2 + d_3^2}{d^2 d_2^2}, \ f_\gamma = \frac{d_1^2 + d_2^2 - d_3^2}{d^2 d_3^2} \ \ \text{and} \ \ f_{spinor} = \frac{2}{d^2} \tag{6.10}$$

We obey (4.2) and achieve a two-parameter deformation of the original weight system at $c_1 = c_2 = c_3$. The corresponding polyhedron, whose edges are built from 4 copies of the vector and 6 copies of the spin or weights, is a (in homogeneously scaled) rhombic dodecahedron with the faces dissected into triangles (Figure 5).

It is important to realize that all examples fulfill the WDVV equation, because the above dimensional reduction argument applies. The crucial properties are the mutual orthogonality of non-concurrent non-parallel edges as well as the incidence relations, which 'sew' the triangles together into a polyhedron. Yet, these properties are only necessary but not sufficient. Finally we remark that all our examples are part of a larger moduli space of $n = 3$ families of WDVV solutions [7,9].

U ≠ 0 Solutions: No-Go 'Theorem' For N > 2

Recall that, for turning on

$$U = -\sum_\alpha g\alpha \ln |\alpha(x)| \ \ \text{With} \ \ \sum_\alpha g_\alpha = C \tag{7.1}$$

In a given F background determined by $\{\alpha, f_\alpha\}$, we need to solve the flatness condition (4.6). In principle, we may modify (4.6) by adding a homogeneous term U_{hom} to the prepotential above, but let us postpone this option for the time being. Then, matching the coefficients of the double poles in (4.6) requires that

$$\text{either } g_\beta = 0 \ \text{ or else } \ \beta \cdot \beta f_\beta = 1 \ \text{ for each convector } \beta \tag{7.2}$$

In the undeformed irreducible root-system solutions, the Weyl group identifies the f_α and g_α coefficients for all roots of the same length. Hence, besides the f_L and f_s values in (5.4) we have couplings g_L and g_s for a number p_L and p_s of long and short positive roots, respectively[b]. This simplifies the trace of (5.3) to

$$n = \sum_\alpha \alpha \cdot \alpha f_\alpha = 2 f_L p_L + \frac{2}{r} f_s p_s \rightarrow p_L + p_s = n \ \text{ if } \ g_L, g_s \neq 0. \tag{7.3}$$

Since the total number $p_L + p_s$ of positive roots always exceeds n (except for $A_1^{\oplus n}$), we are forced to put either $g_s = 0$ or $g_L = 0$. Therefore, all simply-laced root systems are ruled out! For the $r > 1$ root systems, we get

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[b]For explicitness, $p_L = \dfrac{n}{2}\dfrac{rh^\vee - h}{r-1}$ and $p_s = \dfrac{n}{2}\dfrac{r(h-h^\vee)}{r-1}$, with the sum $p = p_L + p_s = \dfrac{n}{2}h$.
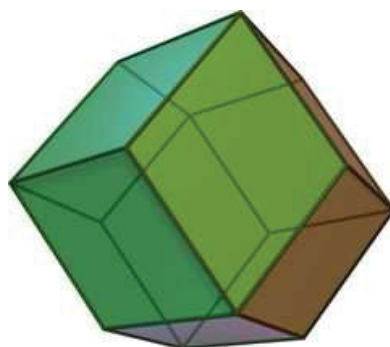

Figure 5: Rhombic dodecahedron.

either $g_s = 0$, $g_L = g \xrightarrow{(7.2)(7.3)} f_s = \dfrac{r}{2}\dfrac{n - p_L}{p_s} \le 0$, $f_L = \dfrac{1}{2}$

(7.4)

or $g_s = g$, $g_L = 0 \xrightarrow{(7.2)(7.3)} f_s = \dfrac{r}{2}$, $f_L = \dfrac{1}{2}\dfrac{n - p_s}{p_L} \le 0.$

(7.5)

We see that in the non-simply-laced one-parameter family (5.4) there is always one member which obeys (7.4) or (7.5). For it, we must still check the remainder of (4.6),

$$\sum_{\alpha \ne \beta} g_\alpha f_\beta \frac{\alpha \cdot \beta}{\alpha(x)\beta(x)} \beta \otimes \beta = 0.$$

(7.6)

Even though its trace is always satisfied, the traceless part is violated for any nontrivial root system with the data (7.4) or (7.5). Hence, there do not exist U solutions of the standard form (7.1) for any Coxeter root system. Perhaps this no-go result may be overcome by adding suitable U_{hom} contributions. Certainly it can be avoided for $n = 2$ because in this case (4.4) may be relaxed (see below). Finally, we have not yet studied the flatness conditions for the deformed root systems of the previous section.

$U \ne 0$ Solutions: Dihedral Solutions for n = 2

As mentioned before, the case of three particles with translation invariance, i.e. $n = 2$, is special for the absence of the WDVV equation. In fact, it is easy to see that any set $\{\alpha\}$ of covectors can be made to obey the left condition in (4.2) with suitably chosen f_α. To study concrete examples, we look at the most symmetric cases, namely the dihedral root systems mentioned earlier.

It is crucial that we take advantage of the freedom at n = 2 to add 'radial terms' in our ansatz:

$$F = -\frac{1}{2}\sum_\alpha f_\alpha \alpha(x)^2 \ln|\alpha(x)| - \frac{1}{2}f_R R^2 \ln R \to \sum_\alpha f_\alpha \alpha \otimes \alpha = (1 - f_R)\mathbb{1},$$

(8.1)

$$U = -\sum_\alpha g_\alpha \ln|\alpha(x)| - g_R \ln R \to \sum_\alpha g_\alpha = C - g_R.$$

(8.2)

The flatness condition then reduces to (7.2) and the trace of (7.6) plus the relation $g_R + (C - g_R)f_R = 0$. It is obeyed for the $I_2(p)$ system if $(g_{even}, g_{odd}) =: (g_S, g_L)$ when p is even and if $g_\alpha =: g \ \forall \alpha$ when p is odd. Turning on g couplings for all covectors fixes $\alpha \cdot \alpha f_\alpha = 1$, and so we obtain

$$p = 2(1 - f_R) \to g_R = \frac{2}{p}C \to \frac{2}{p}C = \sum = \begin{cases} -(g_S \quad g_L) \ \textit{for p even} \\ p\,g \qquad \textit{for p odd} \end{cases}$$

(8.3)

In order to ease the interpretation as three-particle systems, we embed the relative-motion configuration space $\mathbb{R}^2$ into $\mathbb{R}^3 \ni (x^1, x^2, x^3)$ and rotate such that $\alpha_{com} = (1,1,1)$. for identical particles we require invariance under permutations $(x^1, x^2, x^3) \to (x^{\pi 1}, x^{\pi 2}, x^{\pi 3})$ of the full three-body coordinates. This limits p to multiples of 3. The 'radial coordinate' then becomes

$$R^2 = \frac{1}{3} = \{(x^{12})^2 + (x^{23})^2 + (x^{31})^2\} = \frac{2}{3}\{(x^1)^2 + (x^2)^2 + (x^3)^2 - x^1 x^2 - x^2 x^3 - x^3 x^1\}$$

(8.4)

Examples

For illustration we explicitly display the $I_2(p)$ solutions based on (8.1) and (8.2) for the first few values of p.

$\underline{p = 2}$: $A_1 \oplus A_1$ model $f_R = 0$, $g_R = 0$, $\to g_S + g_L = C$

$$\frac{\alpha(x)}{\sqrt{\alpha \cdot \alpha}} \in \left\{\frac{1}{\sqrt{2}}(x^1 - x^2), \ \frac{1}{\sqrt{6}}(x^1 + x^2 - 2x^3)\right\}$$

$$V_B = \frac{g_s^2 + \dfrac{\hbar^2}{4}}{(x^1 - x^2)^2} + \frac{3(g_L^2 + \dfrac{\hbar^2}{4})}{(x^1 + x^2 - 2x^3)^2}$$

$\underline{p=3}$: A_2 model $f_R = -\dfrac{1}{2}$, $g_R = \dfrac{1}{3}C \rightarrow g = \dfrac{2}{9}C$

$$\frac{\alpha(x)}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \frac{1}{\sqrt{2}}(x^1 - x^2),\ \frac{1}{\sqrt{2}}(x^1 - x^3),\ \frac{1}{\sqrt{2}}(x^2 - x^3) \right\}$$

$$V_B = (g^2 + \frac{\hbar^2}{4})\ (\frac{1}{(x^1 - x^2)^2} + \frac{1}{(x^2 - x^3)^2} + \frac{1}{(x^3 - x^1)^2}) + \frac{5}{8}(9g^2 - \hbar^2)\frac{1}{R^2}$$

$\underline{p=4}$: BC_2 model $f_R = -1$, $g_R = \dfrac{1}{2}C \rightarrow g_S + g_L = \dfrac{1}{4}C$

$$\frac{\alpha(x)}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \frac{x^1 - x^2}{\sqrt{2}},\ \frac{x^1 + x^2 - 2x^3}{\sqrt{6}},\ \frac{\tau x^1 + \overline{\tau} x^2 - x^3}{\sqrt{3}},\ \frac{-\overline{\tau} x^1 + \tau x^2 - x^3}{\sqrt{3}} \right\} \text{ With } \left\{ \begin{array}{l} \tau = \dfrac{1}{2}\left(\sqrt{3}+1\right) \\[2mm] \overline{\tau} = \dfrac{1}{2}\left(\sqrt{3}+1\right) \end{array} \right.$$

$$V_B = \frac{g_S^2 + \dfrac{\hbar^2}{4}}{(x^1 - x^2)^2} + \frac{3(g_S^2 + \dfrac{\hbar^2}{4})}{(x^1 + x^2 - 2x^3)^2} + \frac{\dfrac{3}{2}(g_L^2 + \dfrac{\hbar^2}{4})}{(\tau x^1 - \overline{\tau} x^2 - x^3)^2} + \frac{\dfrac{3}{2}(g_L^2 + \dfrac{\hbar^2}{4})}{(\tau x^1 + \overline{\tau} x^2 - x^3)^2} + \frac{6(g_S + g_L)^2 - \dfrac{3}{2}\hbar^2}{R^2}$$

$\underline{p=6}$: G_2 model $f_R = -2$, $g_R = -\dfrac{2}{3}C \rightarrow g_S + g_L = \dfrac{1}{9}C$

$$\frac{\alpha(x)}{\sqrt{\alpha \cdot \beta}} \in \left\{ \frac{x^1 - x^2}{\sqrt{2}},\ \frac{x^1 - x^3}{\sqrt{2}},\ \frac{x^2 - x^3}{\sqrt{2}},\ \frac{2x^1 + x^2 - x^3}{\sqrt{6}},\ \frac{x^1 + x^2 - 2x^3}{\sqrt{6}},\ \frac{-x^1 + 2x^2 - x^3}{\sqrt{6}} \right\}$$

$$V_B = \frac{g_S^2 + \dfrac{\hbar^2}{4}}{(x^1 - x^2)^2} + \frac{3(g_L^2 + \dfrac{\hbar^2}{4})}{(x^1 + x^2 - 2x^3)^2} + \text{Cyclic} + \frac{36(g_S + g_L)^2 - 4\hbar^2}{R^2}$$

$\underline{p=12}$: $I_2(12)$ model $f_R = -5$, $g_R = \dfrac{5}{6}C \rightarrow g_S + g_L = \dfrac{1}{36}C$

$$\frac{\alpha(x)}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \begin{array}{l} \dfrac{x^1 - x^2}{\sqrt{2}}, \dfrac{x^1 - x^3}{\sqrt{2}}, \dfrac{x^2 - x^3}{\sqrt{2}}, \dfrac{2x^1 - x^2 - x^3}{\sqrt{6}}, \dfrac{x^1 + x^2 - 2x^3}{\sqrt{6}}, \dfrac{-x^2 + 2x^2 - x^3}{\sqrt{6}}, \\[3mm] \dfrac{\tau x^1 - x^2 - \overline{\tau} x^3}{\sqrt{3}}, \dfrac{\tau x^1 - \overline{\tau} x^2 - x^3}{\sqrt{3}}, \dfrac{x^1 + \overline{\tau} x^2 - \tau x^3}{\sqrt{3}}, \dfrac{\overline{\tau} x^1 + x^2 - \tau x^3}{\sqrt{3}}, \dfrac{-\overline{\tau} x^1 + \tau x^2 - x^3}{\sqrt{3}}, \dfrac{-x^1 + \tau x^2 - \overline{\tau} x^3}{\sqrt{3}} \end{array} \right\}$$

$$V_B = \frac{g_S^2 + \dfrac{\hbar^2}{4}}{(x^1 - x^2)^2} + \frac{3(g_S^2 + \dfrac{\hbar^2}{4})}{(x^1 + x^2 - 2x^3)^2} + \frac{\dfrac{3}{2}(g_L^2 + \dfrac{\hbar^2}{4})}{(\tau x^1 x^2 - \overline{\tau} x^3)^2} + \frac{\dfrac{3}{2}(g_L^2 + \dfrac{\hbar^2}{4})}{(\tau x^1 - \overline{\tau} x^2 - x^3)^2} + cyclic + \frac{630(g_S + g_L)^2 - \dfrac{35}{2}\hbar^2}{R^2}$$

Finally, let us investigate the effect of adding to U a homogeneous piece U_{hom} for obtaining $U_{tot} = U + U_{\text{hom}}$. At $n = 2$, all we have to solve is the trace of the flatness condition,

$$\partial \cdot \partial U_{\text{hom}} + \sum_{\alpha} f_{\alpha}\alpha \cdot \alpha \frac{\alpha \cdot \partial}{a(x)}U_{\text{hom}} = 0 \text{ besides } x^1 \partial_i U_{\text{hom}} = 0. \tag{9.1}$$

It is convenient to pass to polar coordinates on $\Box^2$ via $(x^1, x^2) = (R\cos\phi, R\sin\phi)$. In the dihedral class $I_2(p)$, the sum over the roots can be performed, and the flatness conditions for U and for U_{hom} is solved by

$$U(R, \phi) = -C \ln R - \left\{ \begin{array}{ll} g\ \ln\left|\cos(p\phi)\right| & \text{for } p \text{ odd} \\[3mm] g_S\ \ln\left|\cos(\dfrac{p}{2}\phi)\right| + g_L\ \ln\left|\sin(\dfrac{p}{2}\phi)\right| & \text{for } p \text{ even} \end{array} \right. \tag{9.1}$$

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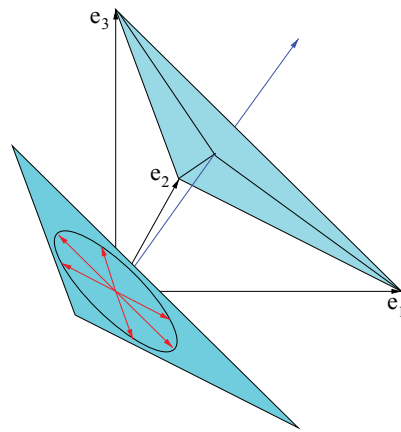**Figure 6:** Embedding of A_2 roots into $\Box^3$.

$$U_{\text{hom}}(\phi) = \frac{\lambda}{p} \ln\left|\tan(\frac{p}{2}\phi + \delta)\right| \ln\left|\tan(\frac{p}{2}\phi + \delta)\right| \text{ with } \delta = \begin{cases} \dfrac{\pi}{4} & \text{for } p \text{ odd} \\ 0 & \text{for } p \text{ even} \end{cases} \tag{9.2}$$

After lifting to the full configuration space $\Box^3$ as in Figure 6, we arrive at

$$\partial_i U_{tot} = -\sum_\alpha g_\alpha \frac{\alpha_i}{\alpha(x)} - \frac{p-2}{p} C \frac{x_i}{R^2} + \lambda \begin{pmatrix} x^2 - x^3 \\ x^3 - x^1 \\ x^1 - x^2 \end{pmatrix} R^{p-2} \prod_\alpha (\alpha(x))^{-1} \tag{9.3}$$

For the A_2 model as the simplest example, one gets

$$F = -\frac{1}{4}\left((x^{12})^2 \ln|x^{12}| + (x^{23})^2 \ln|x^{23}| + (x^{31})^2 \ln|x^{31}|\right) + \frac{1}{4} R^2 \ln R \tag{9.4}$$

$$\vec{\nabla} U_{tot} = [x^{12}x^{23}x^{31}]^{-1} \begin{pmatrix} [\lambda R - g(x^{31} - x^{12})]x^{23} \\ [\lambda R - g(x^{12} - x^{23})]x^{31} \\ [\lambda R - g(x^{23} - x^{31})]x^{12} \end{pmatrix} - \frac{3}{2} g R^{-12} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{9.5}$$

$$V_B^{tot} = (g^2 + \frac{2}{3}\lambda^2 + \frac{\hbar^2}{4})\left(\frac{1}{(x^{12})^2} + \frac{1}{(x^{23})^2} + \frac{1}{(x^{31})^2}\right)$$
$$+ \frac{5}{8}(9g^2 - \hbar^2)R^{-2} + \lambda g R \frac{(x^{12} - x^{23})(x^{23} - x^{31})(x^{31} - x^{12})}{(x^{12}x^{23}x^{31})^2}, \tag{9.6}$$

with $R^2 = \frac{1}{3}\{(x^{12})^2 + (x^{23})^2\}$. A pure Calogero potential is possible only for $g = 0 = \hbar$.

Acknowledgements

References

1. Wyllard N (2000) (Super) conformal many-body quantum mechanics with extended super symmetry. Journal of Mathematical Physics 41.

2. Bellucci S, Galajinsky A, Latini E (2005) New insight into the witten-dijkgraff-verlinde-verlinde equation. Physical Review D 71.

3. Witten E (1990) On the structure of the topological phase of two-dimensional gravity. Nuclear Physics B 340: 281-232.

4. Dijkgraaf R, Verlinde H, Verlinde E (1991) Topological strings in d < 1. Nuclear Physics B 352: 59-86.

5. Martini R, Gragert PKH (1999) Solutions of WDVV equations in seiberg-witten theory from root systems. Journal of Nonlinear Mathematical Physics 6.

6. Veselov AP (1999) Deformations of the root systems and new solutions to generalised WDVV equations. Physics Letter A 261: 297-302.

7. Chalykh A, Veselov AP (2001) Locus configurations and /V-systems*. Physics Letters A 285: 339-349.

8. Galajinsky A, Lechtenfeld O, Polovnikov K (2007) Script N = 4 super conformal calogero models. Journal of High Energy Physics.

9. Feigin MV, Veselov AP (2007) Logarithmic frobenius structures and coxeter discriminants. Advances in Mathematics 212: 143-162.

10. Galajinsky A, Lechtenfeld O, Polovnikov K (2009) N=4 mechanics, WDVV equations and roots. Journal of High Energy Physics.

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